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On theorems of Gelfond and Selberg concerning integral-valued entire functions☆

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Abstract

For each $s \in \mathbb{N}$ define the constant θ_s with the following properties: if an entire function $g(z)$ of type $t(g) < \theta_s$ satisfies

$$g^{(\sigma)}(z) \in \mathbb{Z} \quad \text{for } \sigma = 0, 1, \ldots, s-1 \text{ and } z = 0, 1, 2, \ldots,$$

then g is a polynomial; conversely, for any $\delta > 0$ there exists an entire transcendental function $g(z)$ satisfying the display conditin and $t(g) < \theta_s + \delta$. The result $\theta_1 = \log 2$ is known due to Hardy and Pólya. We provide the upper bound $\theta_s \leqslant \pi s/3$ and improve earlier lower bounds due to Gelfond (1929) and Selberg (1941).

Keywords: Entire function; Integer-valued function; Polynomial interpolation; Group-structure arithmetic method; Selberg integral

0. Introduction and statement of results

The famous theorem, due to Hardy and Pólya, states that *if an entire function* $g(z)$ *of* (*exponential*) *type less than* $\log 2$ *takes integer values at* $z = 0, 1, 2, \ldots$, *then* $g(z)$ *is a*

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polynomial. Clearly, the condition on the type cannot be weakened since *the transcendental function* 2^z *of type* $\log 2$ *is integer-valued for* $z = 0, 1, 2, \dots$. Recall that the type of the entire function $f(z)$ (of order 1) is defined by the formula

$$t(f) := \limsup_{r\to+\infty} \frac{\log |f|_r}{r}, \qquad \text{where} \quad |f|_r := \max_{|z|=r} |f(z)|.$$

The result of Hardy and Pólya was generalized by Gelfond [Ge1] to the case of entire functions taking integer values together with their first $s-1$ derivatives at non-negative integers. A general problem may be regarded as follows: for each $s \in \mathbb{N}$ find the constant $\theta_s > 0$ with the following properties. If an entire function $g(z)$ satisfies

$$g^{(\sigma)}(\mathbb{N}_0) \subset \mathbb{Z} \quad \text{for } \sigma = 0, 1, \dots, s-1 \tag{$*$}$$

and $t(g) < \theta_s$, then g is a polynomial; in opposite, for each $\delta > 0$ there exists an entire *transcendental* function $g(z)$ satisfying $(*)$ and $t(g) < \theta_s + \delta$.

By these means, the Hardy–Pólya theorem asserts $\theta_1 = \log 2$, while Gelfond's theorem in [Ge1] states the estimate

$$\theta_s \geqslant s \log(1 + e^{(1-s)/s}) > s \log(1 + e^{-1}) = s \cdot 0.31326168\dots \quad \text{for } s = 1, 2, \dots. \tag{1}$$

Later, Gelfond's estimate was slightly improved by Selberg [Se],

$$\theta_2 \geqslant \log\left(1 + \sqrt{\frac{4}{e} + \frac{1}{e^2}} + \frac{1}{e}\right) = 0.96907159\dots$$

$$\theta_s > \frac{s}{2} \log\left(1 + \sqrt{\frac{4}{e^2} + \frac{1}{e^4}} + \frac{1}{e^2}\right) = s \cdot 0.31654925\dots \quad \text{for } s = 1, 2, \dots.$$

The crucial ingredient in Selberg's proof was a multidimensional analogue of the Euler beta integral, known now as the Selberg integral [AAR, Chapter 8].

On the other hand, we have never heard of any reasonable upper bound for θ_s when $s > 1$. The aim of our work is to fill the latter gap as well as to improve the earlier (and rather old) estimates of Selberg. Namely, we prove the following two theorems.

Theorem 1. *For each* $s \in \mathbb{N}$ *there exists an entire transcendental function* $g_s(z)$ *satisfying*

(i) $g_s^{(\sigma)}(\mathbb{Z}) \subset \mathbb{Z}$ *for* $\sigma = 0, 1, \dots, s-1$;

(ii) $|g_s|_r \leqslant \exp\left\{s\left(\frac{\pi}{3}r + \frac{1}{2}\log r + c\right)\right\}$ *for each real* $r \geqslant 1$, *where* $c \in \mathbb{R}_+$ *denotes an effectively computable absolute constant*.

As a consequence, one has the upper bound $\theta_s \leqslant \frac{\pi}{3}s$, which is expectedly worse than the known result for $s = 1$: our theorem serves a less general class of entire functions, i.e., satisfying (i) instead of $(*)$.

Remark 1. It should be noted that we may take

$$g_1(z) = \frac{2}{\sqrt{3}} \sin\frac{\pi z}{3} \quad \text{and} \quad g_3(z) = \frac{1}{\pi}\sin \pi z$$

to ensure better estimates than in (ii) in the cases $s = 1$ and 3. But even in the case $s = 2$ one can rather easily check that no simple linear combination of the type

$$a \sin \frac{2\pi z}{3} + b \cos \frac{2\pi z}{3} + c \sin \frac{\pi z}{3} + d \cos \frac{\pi z}{3}$$

with $a, b, c, d \in \mathbb{C}$, not all zero, is good for g_2.

Remark 2. In the assertion of Theorem 1 we can replace "there exists an" by "there exist uncountably many" as we shall indicate in the proof.

Theorem 2. *Let $s \in \mathbb{N}$ and let $g(z)$ be an entire function satisfying $(*)$ and $t(g) < \tilde{\theta}_s$, where*

$$\begin{aligned} &\tilde{\theta}_2 = 0.99407702\ldots, \quad &&\tilde{\theta}_3 = 1.33990538\ldots, \quad &&\tilde{\theta}_4 = 1.67447461\ldots, \\ &\tilde{\theta}_5 = 2.02210976\ldots, \quad &&\tilde{\theta}_6 = 2.36295435\ldots, \quad &&\tilde{\theta}_7 = 2.70097297\ldots, \\ &\tilde{\theta}_8 = 3.04484371\ldots, \quad &&\tilde{\theta}_9 = 3.38570755\ldots. \end{aligned} \tag{2}$$

Then $g(z)$ is a polynomial.

In general, the condition $t(g) < s \cdot 0.32766348$ yields $g(z) \in \mathbb{C}[z]$.

The interpolating technique is the main content in proofs of both theorems, but other ingredients seem to be very different. The proof of Theorem 1 essentially uses ideas from [BS] applied there to an analogous q-problem, while the proof of Theorem 2 exploits the so-called group-structure arithmetic method introduced by Rhin and Viola [RV1,RV2] for proving new bounds of irrationality measures for $\zeta(2)$ and $\zeta(3)$. It is worth mentioning that the arithmetic method allows us to get rid of the Selberg integral.

1. Proof of Theorem 1

1.1. Interpolation

We use ideas from [BS], and choose the following interpolation sequence $(z_\nu)_{\nu=1,2,\ldots}$:

$$\underbrace{0,\ldots,0}_{s \text{ times}},\ \underbrace{1,\ldots,1}_{s \text{ times}},\ \underbrace{-1,\ldots,-1}_{s \text{ times}},\ \underbrace{2,\ldots,2}_{s \text{ times}},\ \ldots,$$

i.e., for any $\nu \in \{(k-1)s+1, \ldots, ks\}$ and $k \in \mathbb{N}$, we have

$$z_\nu = (-1)^k \left\lfloor \frac{k}{2} \right\rfloor, \tag{3}$$

where $\lfloor \cdot \rfloor$ stands for the integer part of a number. Therefore our interpolation polynomials are given by $P_n(z) = \prod_{\nu=1}^{n}(z - z_\nu)$, $n \in \mathbb{N}$; $P_0(z)$ being the constant polynomial 1. With distinct $w_1, \ldots, w_l$ (where $l = l(n) = \lfloor n/s \rfloor$), and exponents $e_1, \ldots, e_l \in \mathbb{N}$ (at least $l-1$ of which equal s) satisfying $e_1 + \cdots + e_l = n$, we have

$$P_n(z) = \prod_{\lambda=1}^{l} (z - w_\lambda)^{e_\lambda}. \tag{4}$$

The idea of this proof is to construct a transcendental function $g(z) = \sum_n B_n P_n(z)$, which is integer-valued at all integers and has small non-zero coefficients B_n.

Let $f(z)$ be an arbitrary entire function. The interpolation coefficients A_{n-1} ($n \in \mathbb{N}$) with respect to the above sequence $(z_\nu)_{\nu\in\mathbb{N}}$ are given by

$$\begin{aligned} A_{n-1} &= \frac{1}{2\pi i}\oint \frac{f(\xi)\,\mathrm{d}\xi}{P_n(\xi)} = \frac{1}{2\pi i}\oint \frac{f(\xi)\,\mathrm{d}\xi}{\prod_{\lambda=1}^{l}(\xi - w_\lambda)^{e_\lambda}} \\ &= \sum_{\lambda=1}^{l}\sum_{\varepsilon_\lambda=0}^{e_\lambda-1}(-1)^{\varepsilon_\lambda}\frac{f^{(e_\lambda-1-\varepsilon_\lambda)}(w_\lambda)}{(e_\lambda-1-\varepsilon_\lambda)!}\sum_{\substack{(\mu_1,\dots,\mu_l)\in\mathbb{N}_0^l\\ \mu_1+\cdots+\mu_l=\varepsilon_\lambda+\mu_\lambda}}\prod_{\substack{\lambda'=1\\ \lambda'\neq\lambda}}^{l}\frac{\binom{e_{\lambda'}+\mu_{\lambda'}-1}{\mu_{\lambda'}}}{(w_\lambda - w_{\lambda'})^{e_{\lambda'}+\mu_{\lambda'}}}, \end{aligned} \tag{5}$$

where the path of integration contains $w_1, \dots, w_l$. Here the right-hand side is a linear form in the n derivatives $f^{(\tau_\lambda)}(w_\lambda)$ with $\lambda \in \{1, \dots, l\}$ and $\tau_\lambda \in \{0, \dots, e_\lambda - 1\}$. Their coefficients are explicitly given rational numbers not depending on f.

From now on, let us suppose $e_1 = \cdots = e_{l-1} = s$ and $e_l \in \{1, \dots, s\}$. The factor of $f^{(e_l-1)}(w_l)$ in (5) is $(e_l-1)!^{-1}\prod_{\lambda'=1}^{l-1}(w_l - w_{\lambda'})^{-s}$ and thus we have

$$\begin{aligned} &(e_l-1)!\prod_{\lambda'=1}^{l-1}(w_l - w_{\lambda'})^{s} A_{n-1} \\ &\quad = \sum_{\lambda=1}^{l-1}\sum_{\mu=0}^{s-1} a_{\lambda,\mu} f^{(s-1-\mu)}(w_\lambda) + \sum_{\mu=1}^{e_l-1} a_{l,\mu} f^{(e_l-1-\mu)}(w_l) + f^{(e_l-1)}(w_l) \end{aligned}$$

with rational $a_{\lambda,\mu}$, again independent of f. Next we inductively define, in the order indicated below, an infinite sequence [1]

$$g_{1,0}, \dots, g_{1,s-1},\ g_{2,0}, \dots, g_{2,s-1},\ \dots,\ g_{l,0}, \dots, g_{l,e_l-1},\ \dots \tag{6}$$

of rational integers by the conditions

$$0 < \sum_{\lambda=1}^{l-1}\sum_{\mu=0}^{s-1} a_{\lambda,\mu} g_{\lambda,s-1-\mu} + \sum_{\mu=1}^{e_l-1} a_{l,\mu} g_{l,e_l-1-\mu} + g_{l,e_l-1} \leqslant 1. \tag{7}$$

Clearly, for $l = 1$, $e_l = 1$ (i.e. $n = 1$) this means $g_{1,0} := 1$. Herewith we put

$$\begin{aligned} B_{n-1} := &\frac{1}{(e_l-1)!\prod_{\lambda=1}^{l-1}(w_l - w_\lambda)^{s}} \\ &\times\left(\sum_{\lambda=1}^{l-1}\sum_{\mu=0}^{s-1} a_{\lambda,\mu} g_{\lambda,s-1-\mu} + \sum_{\mu=1}^{e_l-1} a_{l,\mu} g_{l,e_l-1-\mu} + g_{l,e_l-1}\right) \end{aligned} \tag{8}$$

for each $n \in \mathbb{N}$. In particular, we remark $B_{n-1} \neq 0$ for each $n \in \mathbb{N}$.

---

[1] To see the truth of Remark 2, having chosen all $g_{\lambda,\mu}$ in (6) arising *before* g_{l,e_l-1}, we select g_{l,e_l-1} in such a way that the sum in (7) satisfies $0 < |\text{the sum}| \leqslant 1$. This leads to exactly two distinct choices for g_{l,e_l-1}.

With these B_{n-1} we define

$$g_s(z) := \sum_{n=1}^{\infty} B_{n-1} P_{n-1}(z)$$

and we assert that this g_s is good for our theorem. Having shown that $g_s(z)$ is entire and satisfies (ii), clearly (i) is true as well since $g_s^{(\mu)}(w_\lambda) = g_{\lambda,\mu} \in \mathbb{Z}$ for $\lambda = 1, 2, \ldots$ and $\mu \in \{0, 1, \ldots, s-1\}$. Since no B_{n-1} vanishes, g_s cannot be a polynomial.

To carry out this program we first estimate B_{n-1} from (7), (8) and the postponed Lemma 1, leading to

$$|B_{n-1}| \leqslant \prod_{\lambda=1}^{l-1} |w_l - w_\lambda|^{-s} = (l-1)!^{-s}. \tag{9}$$

Next let k $(= k(n))$ be defined by $2k - 1 < l \leqslant 2k + 1$ or, equivalently, $k := \lfloor l/2 \rfloor$. From our above choice of the interpolation sequence (z_v) and from (4) we deduce

$$P_{n-1}(z) = \prod_{\lambda=1}^{l} (z - w_\lambda)^{e'_\lambda} = (z(z^2-1)\cdots(z^2-(k-1)^2))^s \prod_{\lambda=2k}^{l} \left(z - (-1)^\lambda \left\lfloor \frac{\lambda}{2} \right\rfloor\right)^{e'_\lambda} \tag{10}$$

with $e'_\lambda := e_\lambda$ for $\lambda = 1, \ldots, l-1$, and $e'_l := e_l - 1$, cf. (3).

To get the precise estimate in (ii) we distinguish the two cases: $l = 2k$ and $l = 2k+1$.

Case $l = 2k$: From (9) and (10), using Stirling's formula and the (again postponed) Lemma 2, in the notation

$$\Phi_k(r) := \prod_{j=1}^{k} (r^2 + j^2), \tag{11}$$

we find on $|z| = r$:

$$\begin{aligned} |B_{n-1} P_{n-1}(z)| &< \left(\frac{\sqrt{2k}\, e^{2k}}{\sqrt{2\pi}(2k)^{2k}}\right)^s \Phi_k(r)^s r^s \frac{(r+k)^{e'_{2k}}}{(r^2+k^2)^s} \\ &< \left(\frac{\sqrt{k}}{\sqrt{\pi} 4^k k^{2k}}\right)^s \exp\left\{s\left(k \log(r^2+k^2) + 2r \arctan \frac{k}{r}\right.\right. \\ &\quad \left.\left. +2 + \log\left(1 + \frac{k^2}{r^2}\right)\right)\right\} r^s \frac{(r+k)^{e'_{2k}}}{(r^2+k^2)^s} \\ &= \pi^{-s/2} k^{s/2} \exp\left\{srh\left(\frac{k}{r}\right) + 2s + \log\left(1 + \frac{k^2}{r^2}\right)^s\right\} r^s \frac{(r+k)^{e'_{2k}}}{(r^2+k^2)^s}. \end{aligned} \tag{12}$$

Here the function $h : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$h(t) := t \log(1 + t^{-2}) - t \log 4 + 2 \arctan t.$$

We compute $h'(t) = \log\big((1+t^{-2})/4\big)$ and this expression vanishes in $\mathbb{R}_+$ exactly if $t = 1/\sqrt{3}$. We have $h'(t) > 0$ if $0 < t < 1/\sqrt{3}$ and $h'(t) < 0$ if $t > 1/\sqrt{3}$. Moreover, $h(1/\sqrt{3}) = 2 \arctan(1/\sqrt{3}) = \pi/3$, $h(t) \downarrow 0$ as $t \downarrow 0$, and $h(t) \downarrow -\infty$ as $t \uparrow +\infty$.

Thus, on $|z| = r$, we get from (12)

$$\left|\sum_{\substack{n\in\mathbb{N}\\ k\leqslant 10r}} B_{n-1}P_{n-1}(z)\right| \leqslant \left(\frac{e^2}{\sqrt{\pi}}\right)^s \exp\left(\frac{\pi}{3}sr\right)\frac{1}{r}11^s \sum_{\substack{n\in\mathbb{N}\\ k\leqslant 10r}} k^{s/2}. \tag{13}$$

Since the sum on the right-hand side is less than $(10r)^{1+s/2}s$, where the factor s takes into account that at most s distinct n can lead to the same l (or k, in the case under consideration), inequality (13) yields

$$\left|\sum_{\substack{n\in\mathbb{N}\\ k\leqslant 10r}} B_{n-1}P_{n-1}(z)\right| \leqslant C_1^s r^{s/2} \exp\left(\frac{\pi}{3}sr\right) \tag{14}$$

on $|z| = r$. Clearly, $C_1 > 0$ can be written down explicitly.

We finally have to consider the contribution of those n with $k > 10r$. Starting again from (12) we see

$$\begin{aligned}|B_{n-1}P_{n-1}(z)| &< \left(\frac{e^2}{\sqrt{\pi}}\right)^s k^{s/2} \exp\left\{sk\left(\log\left(1+\left(\frac{r}{k}\right)^2\right) - \log 4 + 2\frac{\arctan(k/r)}{k/r}\right)\right\}\\ &\quad\times\frac{(r+k)^{e'_{2k}}}{r^s}\\ &< \left(\frac{11e^2}{10\sqrt{\pi}}\right)^s k^{3s/2}\exp(-sk),\end{aligned} \tag{15}$$

since

$$\log\left(1+\left(\frac{r}{k}\right)^2\right) - \log 4 + 2\frac{\arctan(k/r)}{k/r} < \log\frac{101}{100} - \log 4 + \frac{\pi}{10} = -1.06218476\ldots$$

for the k's under consideration. Since $k > 10r$ implies $k > 10$ we deduce from (15)

$$|B_{n-1}P_{n-1}(z)| < e^{-sk/2}$$

on $|z| = r$ if n is such that the corresponding k satisfies $k > 10r$. Thus we have

$$\left|\sum_{\substack{n\in\mathbb{N}\\ k>10r}} B_{n-1}P_{n-1}(z)\right| < s\sum_{k>10} e^{-sk/2} < 2e^{-5s}.$$

This combined with (14) yields (ii) in Theorem 1 provided we are in the case $l = 2k$.

The *case* $l = 2k + 1$ will be left to the reader, the arguments being rather similar.

1.2. Postponed lemmas

Here we include two simple lemmas, which we used in the above proof.

Lemma 1. *If* $w_\lambda = (-1)^\lambda\lfloor\lambda/2\rfloor$ *for* $\lambda = 1, 2, \ldots,$ *then*

$$\prod_{\lambda=1}^{l-1}|w_l - w_\lambda| = (l-1)!\,.$$

Proof. From the definition of w_λ we see

$$|w_l - w_{l-2k}| = \left| \left\lfloor \frac{l}{2} \right\rfloor - \left\lfloor \frac{l-2k}{2} \right\rfloor \right| = k$$

for $k = 1, \ldots, \lfloor (l-1)/2 \rfloor$, and

$$|w_l - w_{l+1-2k}| = \left\lfloor \frac{l}{2} \right\rfloor + \left\lfloor \frac{l+1-2k}{2} \right\rfloor = \left\lfloor \frac{l}{2} \right\rfloor + \left\lfloor \frac{l+1}{2} \right\rfloor - k = l - k$$

for $k = 1, \ldots, \lfloor l/2 \rfloor$. Both equalities together imply

$$\begin{aligned} \prod_{\lambda=1}^{l-1} |w_l - w_\lambda| &= \prod_{\substack{\lambda=1 \\ \lambda \equiv l \ (\mathrm{mod}\ 2)}}^{l-1} |w_l - w_\lambda| \prod_{\substack{\lambda=1 \\ \lambda \not\equiv l \ (\mathrm{mod}\ 2)}}^{l-1} |w_l - w_\lambda| \\ &= \left\lfloor \frac{l-1}{2} \right\rfloor ! (l-1) \cdots \left(l - \left\lfloor \frac{l}{2} \right\rfloor \right), \end{aligned}$$

from which our assertion follows. □

The following lemma is a variant of Lemma 2.8 in Welter's dissertation [We]. But whereas Welter uses properties of the Γ-function, our proof leans on simpler arguments, namely just on partial summation.

Lemma 2. *If, for $r \in \mathbb{R}_+$ and $k \in \mathbb{N}$, $\Phi_k(r)$ is defined by* (11), *then one has*

$$\log \Phi_k(r) < k \log(r^2 + k^2) - 2k + 2r \arctan \frac{k}{r} + 2 + \log\left(1 + \left(\frac{k}{r}\right)^2\right).$$

Proof. By partial summation we get

$$\begin{aligned} \log \Phi_k(r) &= \sum_{j=1}^{k} \log(r^2 + j^2) = k \log(r^2 + k^2) - \int_1^k \lfloor t \rfloor \frac{2t \, \mathrm{d}t}{r^2 + t^2} \\ &= k \log(r^2 + k^2) - 2 \int_1^k \frac{t^2 \, \mathrm{d}t}{r^2 + t^2} + 2 \int_1^k \frac{t\{t\} \, \mathrm{d}t}{r^2 + t^2}, \end{aligned}$$

where $\{t\} := t - \lfloor t \rfloor$. Thus,

$$\log \Phi_k(r) = k \log(r^2 + k^2) - 2(k-1) + 2r^2 \int_1^k \frac{\mathrm{d}t}{r^2 + t^2} + 2 \int_1^k \frac{t\{t\} \, \mathrm{d}t}{r^2 + t^2}.$$

Since the first integral is bounded above by $\frac{1}{r} \arctan \frac{k}{r}$, and the second by $\log\bigl(1 + (k/r)^2\bigr)$, we get our inequality as asserted. □

2. Proof of Theorem 2

2.1. Denominator lemma

Let $s \geqslant 2$, and let $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_s$ be non-negative integers satisfying the condition $a_j \leqslant b_k$ for all subscripts $j, k = 1, 2, \ldots, s$. To these numbers we assign the collection $\mathcal{N} = \{b_j - a_k : j, k = 1, 2, \ldots, s\}$, in which all appearances of numbers are counted with their multiplicities.

Define the rational function

$$R(z) = \prod_{j=1}^{s} \frac{(b_j - a_j)!}{(z - a_j)(z - a_j - 1) \cdots (z - b_j)} = \prod_{j=1}^{s} (b_j - a_j)! \frac{\Gamma(z - b_j)}{\Gamma(z - a_j + 1)}$$

and consider its partial-fraction decomposition

$$R(z) = \sum_{k \in \mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{A_{lk}}{(z-k)^l},$$

where $\mathcal{P} = \{\min a_j, \ldots, \max b_j\}$ denotes the set of the poles of $R(z)$ and $\ell(k)$ stands for the order of the pole at $z = k$. By D_n denote the least common multiple of the numbers $1, 2, \ldots, n$ and set $D_0 = 1$ for completeness. The following result is a particular case of [Ne, Proposition 4].

Lemma 3. *Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant \cdots$ be the ordered version of the collection $\mathcal{N}$. Then, for all $k \in \mathcal{P}$ and any integer l with $1 \leqslant l \leqslant \ell(k)$, we have the inclusion*

$$D_{n_1} D_{n_2} \cdots D_{n_{s-1}} A_{lk} \in \mathbb{Z}. \tag{16}$$

Proof. We will show inclusion (16) in more general settings by requiring the parameters $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ to satisfy the inequalities $a_j \leqslant b_j$ for $j = 1, \ldots, s$ only. Clearly $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{s-1} \geqslant 0$, since at least s numbers in $\mathcal{N}$ are non-negative: $b_j - a_j \geqslant 0$ for $j = 1, \ldots, s$. We proceed the proof by induction on the quantity $c = \sum_{j=1}^{s} (b_j - a_j) \geqslant 0$.

The inductive base $c = 0$ corresponds to the case $a_j = b_j$ for all $j = 1, \ldots, s$. We fix $k \in \{a_1, \ldots, a_s\}$ and $l \leqslant \ell(k)$, and assume (by rearranging the subscripts if necessary) that $a_{s-\ell(k)+1} = \cdots = a_s = k$, i.e., $a_j \neq k$ for $j = 1, 2, \ldots, s_0$ with $s_0 = s - \ell(k)$. The standard procedure of determining the partial-fraction coefficients gives

$$\begin{aligned} A_{lk} &= \frac{1}{(\ell(k) - l)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{\ell(k)-l} \left(R(z)(z-k)^{\ell(k)} \right) \Big|_{z=k} \\ &= \frac{1}{(\ell(k) - l)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{\ell(k)-l} \left(\prod_{j=1}^{s_0} \frac{1}{z - a_j} \right) \Bigg|_{z=k} \\ &= (-1)^{s_0} \sum_{l_1 + \cdots + l_{s_0} = \ell(k) - l} \prod_{j=1}^{s_0} \frac{1}{(k - a_j)^{l_j + 1}}. \end{aligned} \tag{17}$$

It remains to note that for all $j=1,\dots,s_0$ we have

$$\frac{1}{(k-a_j)^{l_j+1}}=\frac{1}{\prod_{i=s_0+1}^{s_0+l_j+1}(b_i-a_j)} \quad \text{if } k>a_j=b_j,$$
$$\frac{1}{(k-a_j)^{l_j+1}}=\frac{(-1)^{l_j+1}}{\prod_{i=s_0+1}^{s_0+l_j+1}(b_j-a_i)} \quad \text{if } k<b_j=a_j$$

and the total amount of differences $b_i-a_j, b_j-a_i\in\mathcal{N}$, required for each summand in (17), is equal to $\sum_{j=1}^{s_0}(l_j+1)=\ell(k)-l+s_0=s-l\leqslant s-1$. This proves inclusion (16) in the case $c=0$.

If $c>0$, then the inequality $a_j<b_j$ holds for at least one subscript j, for $j=1$, say. Multiplying both sides of the identity

$$1=\frac{z-a_1}{b_1-a_1}-\frac{z-b_1}{b_1-a_1}$$

by $R(z)$, we obtain the relation $R(z)=R(a_1+1;z)-R(b_1-1;z)$, where the records a_1+1 and b_1-1 mean the changes of the corresponding parameters only. It can be easily seen that the numbers in the collections $\mathcal{N}$ for the rational functions on the left-hand side of the relation do not exceed the corresponding numbers in the collection $\mathcal{N}$ for the right-hand side, but the value of c for $R(a_1+1;z)$ and $R(b_1-1;z)$ is by 1 less than for $R(z)$. Therefore, we may apply the inductive step arguments to arrive at (16), and the lemma follows. □

The following fact will be rather important to us: the collection $\mathcal{N}$ and the collection $\{n_1,n_2,\dots,n_{s-1}\}$ of its $s-1$ successive maxima are invariant under any rearrangement of the parameters in the group $b_1,b_2,\dots,b_s$ (and/or in the group $a_1,a_2,\dots,a_s$).

2.2. Settings

General shapes of interpolation polynomials are as follows (cf. Section 1):

$$Q_n(z)=\prod_{j=1}^{s}(z-a_j)(z-a_j-1)\cdots(z-b_j), \qquad Q_n(z)\mid Q_{n+1}(z),$$

where $\deg Q_n=n$ and all a_j's and b_j's are rational integers. In [Se, Hilfsatz II], it is shown that if $b_j=O(n)$ as $n\to\infty$ and the interpolation coefficients

$$A_n=\frac{1}{2\pi i}\oint\frac{g(z)}{Q_n(z)}\,\mathrm{d}z$$

vanish for all $n\geqslant n_0$, then $g(z)$ is a polynomial. Moreover, it is sufficient to prove that $A_{n_\nu}=0$ for all $\nu\geqslant\nu_0$, where the subsequence $\{n_\nu\}_{\nu=0,1,\dots}\subset\mathbb{N}_0$ is sufficiently dense, namely, $0<n_{\nu+1}-n_\nu\leqslant\text{const}$. (Indeed, all analytic estimates for interpolation coefficients, like (19) below, have such form that if $|A_{n_\nu}|\leqslant C$, then $|A_n|\leqslant C$ for all $n\leqslant n_\nu$.)

Let n be an increasing parameter in the construction below. We fix the tuple of parameters $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)$ and $\boldsymbol{\beta}=(\beta_1,\dots,\beta_s)$ satisfying the condition

$$\beta_j\geqslant\alpha_k\geqslant0 \qquad \text{for all} \quad 1\leqslant j,k\leqslant s$$

and take

$$a_j = \alpha_j n, \quad b_j = \beta_j n, \qquad j = 1, 2, \ldots, s.$$

In these settings the total degree of the polynomial

$$Q_n(z) = \prod_{j=1}^{s} (z - a_j)(z - a_j - 1) \cdots (z - b_j)$$

is $\sum_{j=1}^{s}(b_j - a_j + 1) = n\sum_{j=1}^{s}(\beta_j - \alpha_j) + s$, but since gaps of length $\sum_{j=1}^{s}(\beta_j - \alpha_j)$ are allowed, it is enough to show that

$$B_n = \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{g(z)}{Q_n(z)} \, \mathrm{d}z$$

vanishes for $n \geqslant n_0$ (that implies $g(z) \in \mathbb{C}[z]$). Here Γ_n denotes a contour with interior including all zeros of the polynomial $Q_n(z)$.

2.3. *Arithmetic part*

In order to apply Lemma 3, take $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_{s-1}$ to be the first $s-1$ successive maxima in the collection $\mathcal{N} = \{\beta_j - \alpha_k : 1 \leqslant j, k \leqslant s\}$ and set $\Delta_n = \prod_{j=1}^{s-1} D_{v_j n}$. If

$$\Delta_n \frac{\prod_{j=1}^{s}((\beta_j - \alpha_j)n)!}{Q_n(z)} = \sum_{k \in \mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{A_{lk}}{(z-k)^l},$$

then all A_{lk} are integers by Lemma 3. For any permutation σ of the set $\{1, \ldots, m\}$, we set

$$\Pi(\sigma) = \prod_{j=1}^{s} ((\beta_{\sigma(j)} - \alpha_j)n)!$$

and use the group-structure arithmetic method in the following manner. Again from Lemma 3 and due to the symmetry of our construction it follows that, for any σ, the coefficients $A_{lk}^{(\sigma)}$ in the decomposition

$$\sum_{k \in \mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{A_{lk}^{(\sigma)}}{(z-k)^l} = \Delta_n \frac{\Pi(\sigma)}{Q_n(z)} = \frac{\Pi(\sigma)}{\Pi(\mathrm{id})} \sum_{k \in \mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{A_{lk}}{(z-k)^l}$$

are integers. Therefore, if for each prime p,

$$\omega_p = \max_{\sigma} \left\{ \mathrm{ord}_p \frac{\Pi(\mathrm{id})}{\Pi(\sigma)} \right\} \geqslant 0$$

and $\Omega_n = \prod_p p^{\omega_p}$, then the coefficients $A'_{lk} = A_{lk}\Omega_n^{-1}$ in the decomposition

$$\Delta_n \Omega_n^{-1} \frac{\prod_{j=1}^{s}((\beta_j - \alpha_j)n)!}{Q_n(z)} = \sum_{k \in \mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{A'_{lk}}{(z-k)^l}$$

are all integers. For primes $p > \sqrt{Cn}$ (where $C = \max \beta_j$, say) the procedure of algorithmic determining ω_p is known: take

$$\omega(x) = \max_{\sigma}\left(\sum_{j=1}^{s} \lfloor(\beta_j - \alpha_j)x\rfloor - \sum_{j=1}^{s} \lfloor(\beta_{\sigma(j)} - \alpha_j)x\rfloor\right); \tag{18}$$

then $\omega_p = \omega(n/p)$ (since $\mathrm{ord}_p N! = \lfloor N/p\rfloor$ for any prime $p > \sqrt{N}$). The function $\omega(x)$ is 1-periodic and by application of the Chudnovsky–Rukhadze–Hata arithmetic scheme (see, e.g., [Zu, Lemma 4.4]), we get

$$\lim_{n\to\infty} \frac{\log\, \Omega_n}{n} = \int_0^1 \omega(x)\, \mathrm{d}\psi(x),$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. On the other hand, the prime number theorem yields

$$\lim_{n\to\infty} \frac{\log\, D_{\nu_j n}}{n} = \nu_j, \qquad j = 1, 2, \ldots, s-1.$$

Following the lines of the proof of Hilfsatz V in [Se, p. 166], we see that the numbers

$$\begin{aligned} B_n' &= B_n (s-1)!\Delta_n \Omega_n^{-1} \prod_{j=1}^{s} ((\beta_j - \alpha_j)n)! \\ &= \frac{1}{2\pi i}\oint_{\Gamma_n} g(z) \sum_{k\in\mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{(s-1)! A_{lk}'}{(z-k)^l}\,\mathrm{d}z = \sum_{k\in\mathcal{P}} \sum_{l=1}^{\ell(k)} \frac{(s-1)!}{(l-1)!} A_{lk}' g^{(l-1)}(k) \end{aligned}$$

are all integers since $\ell(k) \leqslant s$ for each $k \in \mathcal{P}$.

2.4. *Some 'complex' analysis*

Take $\Gamma_n = \{z : |z| = \gamma n\}$ for some constant $\gamma \geqslant \gamma_0 = \max\, \beta_j > 0$. Then

$$|B_n| \leqslant \frac{\gamma n}{2\pi} \int_{-\pi}^{\pi} \frac{|g(z)|}{Q_n(z)}\,\mathrm{d}\varphi \leqslant \frac{C_1 n e^{\theta\gamma n}}{Q_n(\gamma n)}, \tag{19}$$

where θ denotes the type of the entire function $g(z)$ (i.e., $|g(z)| < Ce^{\theta|z|}$). By Stirling's asymptotic formula, we have

$$\begin{aligned} &\frac{((\beta_j - \alpha_j)n)!}{(\gamma n - \alpha_j n)(\gamma n - \alpha_j n - 1)\cdots(\gamma n - \beta_j n)} = \frac{\Gamma((\gamma - \beta_j)n)\,\Gamma((\beta_j - \alpha_j)n + 1)}{\Gamma((\gamma - \alpha_j)n + 1)} \\ &\quad\sim C_2(\alpha_j, \beta_j, \gamma) n^{-1/2} \left(\frac{(\gamma - \beta_j)^{\gamma - \beta_j}(\beta_j - \alpha_j)^{\beta_j - \alpha_j}}{(\gamma - \alpha_j)^{\gamma - \alpha_j}}\right)^n, \qquad j = 1, \ldots, s, \end{aligned}$$

as $n \to \infty$. Finally,

$$\limsup_{n\to\infty} \frac{\log |B_n'|}{n} \leqslant \varkappa = (\nu_1 + \cdots + \nu_{s-1}) - \int_0^1 \omega(x)\,\mathrm{d}\psi(x) + \min_{x \geqslant \gamma_0} f(x),$$

where

$$f(x) = \theta x + \sum_{j=1}^{s} ((x-\beta_j)\log(x-\beta_j) - (x-\alpha_j)\log(x-\alpha_j) + (\beta_j-\alpha_j)\log(\beta_j-\alpha_j)). \quad (20)$$

If $\varkappa < 0$, we automatically obtain $B'_n = 0$ (since $B'_n \in \mathbb{Z}$) and hence $B_n = 0$ for all sufficiently large n. The minimum of the function $f(x)$ is achieved at the point $x = x_0$, which satisfies $f'(x_0) = 0$ with

$$f'(x) = \theta + \sum_{j=1}^{s} (\log(x-\beta_j) - \log(x-\alpha_j)).$$

For this point $x = x_0$ we obtain

$$\min_{x \geqslant \gamma_0} f(x) = f(x_0) = f_0(x_0),$$

where

$$\begin{aligned} f_0(x) &= f(x) - xf'(x) \\ &= \sum_{j=1}^{s} (\alpha_j \log(x-\alpha_j) - \beta_j \log(x-\beta_j) + (\beta_j-\alpha_j)\log(\beta_j-\alpha_j)). \end{aligned} \quad (21)$$

Since

$$f'_0(x) = \sum_{j=1}^{s} \left(\frac{\alpha_j}{x-\alpha_j} - \frac{\beta_j}{x-\beta_j} \right) = -\sum_{j=1}^{s} \frac{(\beta_j-\alpha_j)x}{(x-\alpha_j)(x-\beta_j)} < 0$$

for $x \geqslant \gamma_0$, the function $f_0(x)$ decreases for $x \geqslant \gamma_0$. Suppose that we determine the (unique) point $x = x_1 > \gamma_0$ such that

$$f_0(x_1) = -(v_1 + \cdots + v_{s-1}) + \int_0^1 \omega(x)\,\mathrm{d}\psi(x).$$

Then taking

$$\tilde{\theta} = -\sum_{j=1}^{s} (\log(x_1-\beta_j) - \log(x_1-\alpha_j)),$$

we obtain that the condition $\theta < \tilde{\theta}$ yields $\varkappa < 0$.

2.5. *Proof of Theorem 2*

Applying the above scheme for the cases $s = 2, 3, \ldots, 9$, we get the values $\tilde{\theta}$ in (2) corresponding to the following (optimal) tuples of the parameters:

$$\begin{aligned}
s = 2: &\quad (\boldsymbol{\alpha}; \boldsymbol{\beta}) = (0, 1; 9, 10), \\
s = 3: &\quad (\boldsymbol{\alpha}; \boldsymbol{\beta}) = (0, 1, 2; 11, 12, 13), \\
s = 4: &\quad (\boldsymbol{\alpha}; \boldsymbol{\beta}) = (0, 1, 2, 3; 17, 18, 19, 20), \\
s = 5: &\quad (\boldsymbol{\alpha}; \boldsymbol{\beta}) = (0, 1, 2, 3, 4; 22, 23, 24, 25, 26),
\end{aligned}$$

$$\begin{aligned} s=6: &\quad (\boldsymbol{\alpha};\boldsymbol{\beta})=(0,1,2,3,4,5;27,28,29,30,31,32),\\ s=7: &\quad (\boldsymbol{\alpha};\boldsymbol{\beta})=(0,1,\dots,6;33,34,\dots,39),\\ s=8: &\quad (\boldsymbol{\alpha};\boldsymbol{\beta})=(0,1,\dots,7;38,39,\dots,45),\\ s=9: &\quad (\boldsymbol{\alpha};\boldsymbol{\beta})=(0,1,\dots,8;42,43,\dots,50). \end{aligned}$$

Also note that Gelfond's estimate (1) corresponds to the choice $(\boldsymbol{\alpha};\boldsymbol{\beta})=(0,\dots,0;1,\dots,1)$ in our notation.

To proceed with the second (general in s) assertion of Theorem 2, fix the collection $(\boldsymbol{\alpha}^*;\boldsymbol{\beta}^*)=(\alpha_1^*,\dots,\alpha_m^*;\beta_1^*,\dots,\beta_m^*)$ for some $m\geqslant 2$, with the additional restrictions

$$\begin{aligned} &\alpha_1^*\leqslant\cdots\leqslant\alpha_m^*<\beta_1^*\leqslant\cdots\leqslant\beta_m^*,\\ &\beta_1^*-\alpha_1^*=\cdots=\beta_m^*-\alpha_m^*=\mu^* \end{aligned} \tag{22}$$

to simplify the general consideration. Set $\nu^*=\beta_m^*-\alpha_1^*$, the functions $f^*(x)$ and $f_0^*(x)$ defined in (20) and (21) (with θ replaced by θ^*) for the collection $(\boldsymbol{\alpha}^*;\boldsymbol{\beta}^*)$, and compute the arithmetic functions $\omega_l^*(x)$ in (18) for the cut collections $(\alpha_1^*,\dots,\alpha_l^*;\beta_1^*,\dots,\beta_l^*)$, $l=0,1,\dots,m$, respectively (so that $\omega_0^*(x)$ and $\omega_1^*(x)$ are identically zero) together with the corresponding arithmetic contributions

$$I_l^*=\int_0^1\omega_l^*(x)\,\mathrm{d}\psi(x),\qquad l=0,1,\dots,m.$$

Assume also the condition

$$\nu^*\geqslant\frac{l}{m}I_m^*-I_l^*,\qquad l=0,1,\dots,m \tag{23}$$

(for $l=0$ it clearly holds).

Our (close to optimal) choice of the collection $(\boldsymbol{\alpha};\boldsymbol{\beta})$ for any $s\geqslant 2$ is as follows:

$$\alpha_j=\alpha^*_{j\ (\mathrm{mod}\,m)},\qquad \beta_j=\beta^*_{j\ (\mathrm{mod}\,m)}\qquad \text{for } j=1,\dots,s.$$

Write $s=km+l$, where $0\leqslant l\leqslant m-1$. Clearly, we get $\nu_j\leqslant\nu^*$ for $j=1,\dots,s-1$ and $\omega(x)\geqslant k\omega_m^*(x)+\omega_l^*(x)$, and by (23)

$$(\nu_1+\cdots+\nu_{s-1})-\int_0^1\omega(x)\,\mathrm{d}\psi(x)\leqslant(s-1)\nu^*-(kI_m^*+I_l^*)\leqslant\frac{s}{m}(m\nu^*-I_m^*). \tag{24}$$

Denote by $x_1^*>\gamma_0=\max\beta_j^*$ the unique solution of the equation $f_0^*(x)=-(m\nu^*-I_m^*)$ and set

$$\tilde\theta^*=-\sum_{j=1}^m\bigl(\log(x_1^*-\beta_j^*)-\log(x_1^*-\alpha_j^*)\bigr).$$

As we have already seen, the condition $\theta^*<\tilde\theta^*$ implies

$$\varkappa^*:=m\nu^*-I_m^*+\min_{x\geqslant\gamma_0}f^*(x)<0. \tag{25}$$

Restrictions (22) imply that for any real $x > \gamma_0$ the sequence of the m real numbers

$$h_j(x) = (x-\beta_j^*)\log(x-\beta_j^*) - (x-\alpha_j^*)\log(x-\alpha_j^*) + (\beta_j^*-\alpha_j^*)\log(\beta_j^*-\alpha_j^*)$$

increases[2] with $j = 1, \ldots, m$. Therefore,

$$\sum_{j=1}^{l} h_j(x) \leqslant \frac{l}{m}\sum_{j=1}^{m} h_j(x) = \frac{l}{m}\bigl(-\theta^* x + f^*(x)\bigr) \quad \text{for } x > \gamma_0$$

and, as a corollary,

$$f(x) \leqslant \frac{s}{m} f^*(x) \qquad \text{for } x > \gamma_0, \tag{26}$$

provided that

$$\theta \leqslant \frac{s}{m}\theta^*. \tag{27}$$

Lemma 4. *If two real functions $g_1(x)$ and $g_2(x)$ satisfy $g_1(x) \leqslant g_2(x)$ for $x \in X \subset \mathbb{R}$ and both functions admit their minima on X, then*

$$\min_{x \in X} g_1(x) \leqslant \min_{x \in X} g_2(x).$$

We omit the proof of this clear observation and write the following consequence of it and relation (26):

$$\min_{x \geqslant \gamma_0} f(x) \leqslant \frac{s}{m} \min_{x \geqslant \gamma_0} f^*(x),$$

hence by (24)

$$\varkappa = (v_1 + \cdots + v_{s-1}) - \int_0^1 \omega(x)\,\mathrm{d}\psi(x) + \min_{x \geqslant \gamma_0} f(x) \leqslant \frac{s}{m}\varkappa^* \tag{28}$$

provided that (27) holds. Finally, from (25) and (28) we obtain that if $\theta \leqslant \frac{s}{m}\tilde{\theta}^*$, then $\varkappa < 0$ and hence $g(z)$ with $t(g) = \theta \leqslant \frac{s}{m}\tilde{\theta}^*$ should be a polynomial.

The choice $(\alpha^*, \beta^*) = (0, 1, 2, 3, 4; 39, 40, 41, 42, 43)$ gives $v^* = 43$ and $\frac{1}{5}\tilde{\theta}^* = 0.32766348\ldots$. This completes the proof of Theorem 2.

3. Concluding remarks

Problems similar to those considered in this work are also known in the case of entire functions taking integer values with their derivatives at the points $z = q^n$, $n = 0, 1, 2, \ldots$; $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ is fixed. The corresponding estimates from both below and above for a

[2] *Hint*: prove that the function $h(\beta) = (x-\beta)\log(x-\beta) - (x-\beta+\mu^*)\log(x-\beta+\mu^*)$ increases with β changing from 0 to x; real x is fixed.

q-analogue of the constant $\tilde{\theta}_s$ were first established by Gelfond in [Ge2]. Later, the upper bound was considerably improved in [BS]. However, no results sharpening the lower bound appeared, and we would like to conclude this paper by saying that the q-analogue of the arithmetic method used in the proof of Theorem 2 (including Selberg's method in [Se] as a particular case) does not allow one to improve this lower bound.

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